A direct approach to design of geometry and forces of tensegrity systems

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Abstract

In the process of designing a tensegrity system, some constraints are usually introduced for geometry and/or forces to ensure uniqueness of the solution, because the tensegrity systems are underdetermined in most cases. In this paper, a new approach is presented to enable designers to specify independent sets of axial forces and nodal coordinates consecutively, under the equilibrium conditions and the given constraints, to satisfy the distinctly different requirements of architects and structural engineers. The proposed method can be used very efficiently for practical applications because only linear algebraic equations are to be solved, and no equation of kinematics or material property is needed. Some numerical examples are given to show not only efficiency of the proposed method but also its ability of searching new configurations.
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1. Introduction

Tensegrity is an acronym, a contraction of *tensional integrity*, named by Fuller (1975). A strict tensegrity is composed of a set of continuous cables in tension, and a set of discontinuous struts in compression. However, many structures developed nowadays from this basic idea do not fit the definition exactly. According to the mechanical characteristics of each type of tensegrity system, Motro (1996) classified them into three

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classes, which are (a) self-stressing systems (also, degenerated tensegrity systems), (b) prestressed systems (also, tensegric domes or cable domes), (c) self-stressed systems (also, true tensegrity systems).

Haber and Abel (1982) concluded that the basic parameters involved in the design of tension membrane structures, also suitable for the case of tensegrity systems, are: (a) topology, (b) internal forces, (c) external loads, (d) configuration, and (e) geometrical constraints. Among these, topology defines the connectivity of members and nodes. In the design of tensegrity systems, members are assumed to be pin-jointed and external loads are applied at nodes if exist. So the members transmit only axial forces, either in tension or compression. The geometrical configuration is described in terms of nodal coordinates. Determination of geometrical configuration at the equilibrium state is known as *form-finding* or *shape-finding* that is considered a key step in the design of tensegrity systems. Geometrical constraints, such as locations and directions of members and symmetry properties, are often needed to ensure a unique solution of the form-finding problem because most of the tensegrity systems are underdetermined; i.e. the member forces and nodal locations cannot be determined uniquely only from the equilibrium conditions.

There have been extensive researches on analysis and design of tensegrity systems, such as Hanaor (1988), Jager and Skelton (2004), Motro et al. (1986), Pellegrino (1990), Pellegrino and Calladine (1986) and Sultan et al. (2001). Many methods have also been proposed for the form-finding problem. Some of them, such as dynamic relaxation by Barnes (1999) and Motro (1984) and force density method by Schek (1974) and Vassart and Motro (1999), have been originally used for the form-finding problem of membrane structures and cable nets and then extended to tensegrity systems.

Tibert and Pellegrino (2003) classified the existing methods for the form-finding of tensegrity system into two categories; i.e. kinematical methods and statical methods. The kinematical methods determine the configuration of either maximum total length of the struts or minimum total length of the cable elements, while the total length of the elements of the other type is kept constant. The statical methods search for equilibrium configurations that allow existence of a state of prestress in the structure with given topology and forces, where the force density method can be effectively used.

For the design of tensegrity systems in architectural application, it is desirable for architects that the characteristics of configuration, such as directions and locations of members or locations of supports, can be specified, while structural engineers are concerned with its mechanical properties defined by the member forces. Hence, the goal of us in this study is to present an effective approach for the form-finding problem of tensegrity systems for the purpose of satisfying both of these requirements simultaneously.

Ohsaki and Kanno (2003) pointed out that the set of member forces cannot be specified arbitrarily because the equilibrium conditions and geometrical constraints may be violated. For a tensegrity with fixed configuration, the number of independent member force vectors at self-equilibrium can be easily found by investigating the rank of the equilibrium matrix. An efficient approach, which can directly find the number of independent member directions (force vectors) and nodal locations of tensegrity system with given topology, is strongly desired.

In this paper, a new approach is presented for specifying independent set of axial forces and then nodal coordinates consecutively, under equilibrium conditions and geometrical constraints. The incidence matrix of a directed graph (Harary, 1969; Kaveh, 2004) is used in the formulation of the equilibrium conditions. The components of the member force vector are taken as the variables (Williamson and Skelton, 2003) at the first stage, and then the nodal coordinates at the second stage of the approach.

The effectiveness of the proposed method can be summarized as follows:

(a) The directions of some members can be specified directly based on the designer's preference.
(b) The numbers of components of the generalized force vector and the nodal coordinate vector to be specified can be systematically obtained.
(c) The equilibrium conditions in large-deformation range can be formulated as a set of linear equations in terms of generalized force vector.

(d) The method presents a general procedure for checking the existence of solution to a form-finding problem.

The paper is organized as follows: Section 2 describes the equilibrium conditions of the structures by using graph theory. In Section 3, the geometrical constraints are formulated to represent designer's preferences, including (a) member directions, (b) member locations, and (c) symmetry properties, in terms of axial force vector. Section 4 describes the proposed form-finding approach in detail, and some numerical examples are presented in Section 5 to illustrate the validity of the proposed method for generating various shapes. Finally, conclusions are given in Section 6.

2. Equilibrium conditions

In this study, the following assumptions are adopted to simplify the problem:

(a) The topology of the structure, which is defined by the designer, is known.
(b) Members are connected by pin joints.
(c) No external load is applied and the self-weight of the structure is neglected during the design procedure.

Since the incidence matrix of a directed graph is used for formulating the equilibrium equations in terms of the member force vectors, we start with a brief description of the graph theory as described by Harary (1969). The words *edge* and *node* in graph theory are called *member* with direction and *node* in this paper, respectively. Let m denote the number of members including cables and struts. The number of nodes including the supports of a structure is denoted by n.

Suppose that member k is connected to nodes i and j, and directed from node i to j ($i < j$). The incidence matrix $\overline{\mathbf{B}} = [\overline{B}_{(k,p)}] \in \Re^{m \times n}$ of the structure regarded as a directed graph can be defined as

$$\overline{B}_{(k,p)} = \begin{cases} 1 & \text{for } p = i \\ -1 & \text{for } p = j \\ 0 & \text{in other cases} \end{cases}$$

By using $\overline{\mathbf{B}}$, the equilibrium matrix $\mathbf{B}$ is defined as

$$\mathbf{B} = \overline{\mathbf{B}} \otimes \mathbf{I}$$

where $\otimes$ and $\mathbf{I} \in \Re^{s \times s}$ denote tensor product and identity matrix, respectively, and s is the number of dimension of the space; i.e. $s = 2$ or 3.

Let $\mathbf{v}_k \in \Re^s$ denote the axial force vector of member k that is connected to nodes i and j ($i < j$). We define the positive direction, i.e., tensile state, of $\mathbf{v}_k$ as shown in Fig. 1(a), starting from i and directing to j. Fig. 1(b) shows the negative direction of $\mathbf{v}_k$, which means that member k is in compression.
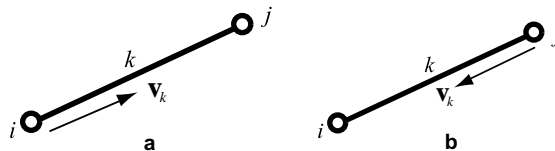

Fig. 1. Definition of axial force vector $\mathbf{v}_k$ ($i < j$): (a) positive (in tension) and (b) negative (in compression).

For member k of a two-dimensional structure, its force components of $\mathbf{v}_k$ are written as $\mathbf{v}_k = (v_k^x, v_k^y)^\top$. In three-dimensional space, $\mathbf{v}_k = (v_k^x, v_k^y, v_k^z)^\top$. The axial force vector of all members, called *generalized force vector*, is defined as $\mathbf{v} = (\mathbf{v}_1^\top, \ldots, \mathbf{v}_m^\top)^\top \in \Re^{sm}$.

Fig. 2 shows a two-dimensional prestressed tensegrity system with $n = 6$ and $m = 8$. Nodes 5 and 6 are the supports that are called *fixed nodes* in this paper, because their locations are fixed. The equilibrium matrix $\mathbf{B} \in \Re^{16 \times 12}$ can be divided into sub-matrices $\mathbf{B}^a$ and $\mathbf{B}^b$ that correspond to the equilibrium matrices of free nodes and fixed nodes, respectively, as

$$
\mathbf{B} = (\mathbf{B}^a\ \mathbf{B}^b) =
\begin{array}{c|cccccccc|cccc}
 & n_1^x & n_1^y & n_2^x & n_2^y & n_3^x & n_3^y & n_4^x & n_4^y & n_5^x & n_5^y & n_6^x & n_6^y \\
\hline
m_1^x & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
m_1^y & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
m_2^x & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_2^y & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_3^x & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
m_3^y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
m_4^x & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_4^y & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_5^x & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
m_5^y & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
m_6^x & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
m_6^y & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
m_7^x & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
m_7^y & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
m_8^x & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
m_8^y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\end{array}
$$

Note that n_i^x and n_i^y denote the columns corresponding to the equilibrium in x- and y-directions, respectively, at node i. m_k^x and m_k^y denote the rows corresponding to the x- and y-components, respectively, of member k.

Since all the free nodes are in self-equilibrium state, the equilibrium equations of a free node, e.g. node 4 in Fig. 2, are written as

$$
\begin{aligned}
-v_5^x - v_7^x + v_8^x = 0 \\
-v_5^y - v_7^y + v_8^y = 0
\end{aligned}
\tag{1}
$$

Eq. (1) can be rewritten by using the equilibrium matrix as

$$
(\mathbf{B}^\top)_7 \mathbf{v} = 0
$$
$$
(\mathbf{B}^\top)_8 \mathbf{v} = 0
$$

where $(\mathbf{B}^\top)_i$ denotes the ith row of $\mathbf{B}^\top$. $(\mathbf{B}^\top)_7$ and $(\mathbf{B}^\top)_8$ correspond to the x- and y-directions, respectively, of node 4. So the global equilibrium condition of all the free nodes in terms of unknown generalized axial force vector $\mathbf{v}$ can be written as

$$
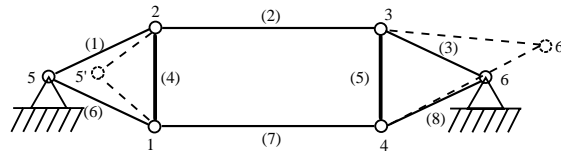\mathbf{B}^{a\top} \mathbf{v} = \mathbf{0}
\tag{2}
$$

Fig. 2. A two-dimensional tensegrity system.

In designing tensegrity systems with supports, the locations of the supports should be specified. In Section 3, a direct approach is presented for specifying force vectors and member directions. In the proposed approach, some components of force vectors are first specified. However, if we specify a set of axial force vectors arbitrarily satisfying Eq. (2), the form-finding process may end up with an undesirable configuration as shown in the dotted lines in Fig. 2; i.e. nodes 5 and 6 are located unfavorably, because the locations of the fixed nodes (supports in this example) have not been incorporated in Eq. (2).

To present a unified approach to shape design of structures with and without supports, we introduce auxiliary members called *fixed members* to connect the supports. For the structure shown in Fig. 2, we connect nodes 5 and 6 with the auxiliary member 9 as shown in Fig. 3. The locations of the auxiliary members are to be fixed in Section 3 to transform the supported tensegrity into a self-stressing or self-stressed system.

Let m_f and n_f denote the numbers of fixed members and nodes, respectively, where $m_f = 1$ and $n_f = 2$ for the structure in Fig. 3. Note that the members and nodes are numbered such that the nodes $\{1, \ldots, n - n_f\}$ and members $\{1, \ldots, m - m_f\}$ correspond to free nodes and members, respectively. So $\mathbf{B}^f \in \mathfrak{R}^{sm_f \times sn_f}$ corresponding to fixed members and nodes is placed in the lower-right of $\mathbf{B}$ as

$$
\mathbf{B} = \left(\begin{array}{c|c} \mathbf{B}^a & \mathbf{B}^b \\ \hline \mathbf{O} & \mathbf{B}^f \end{array} \right) =
\begin{array}{c}
\\ m_1^x \\ m_1^y \\ m_2^x \\ m_2^y \\ m_3^x \\ m_3^y \\ m_4^x \\ m_4^y \\ m_5^x \\ m_5^y \\ m_6^x \\ m_6^y \\ m_7^x \\ m_7^y \\ m_8^x \\ m_8^y \\ \hline m_9^x \\ m_9^y
\end{array}
\begin{array}{|cccccccc|cccc|}
n_1^x & n_1^y & n_2^x & n_2^y & n_3^x & n_3^y & n_4^x & n_4^y & n_5^x & n_5^y & n_6^x & n_6^y \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\end{array}
$$

This way, all the fixed nodes are converted to free nodes, and the equilibrium equation of the structure can be written as
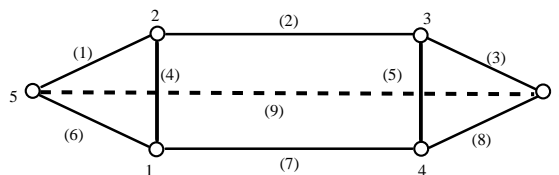
$$\mathbf{B}^\top \mathbf{v} = \mathbf{0} \tag{3}$$

Fig. 3. The two-dimensional tensegrity system with fixed members.

3. Geometrical constraints in terms of generalized force vector

3.1. Member directions

In the design process of tensegrity systems, it is desirable that the directions of some members can be directly specified. The direction of a member, however, should coincide with that of its force vector, because members of a tensegrity system can transmit only axial forces.

Consider a tensegrity system in three-dimensional space, and let $\mathbf{e}_k = (e_k^x, e_k^y, e_k^z)^\top$ denote a vector in the direction of member k, where the vector $\mathbf{e}_k$ of some members are given by designer's preference. The direction vector $\mathbf{e}_k$ and the force vector $\mathbf{v}_k$ of member k should satisfy the relation $\mathbf{e}_k \times \mathbf{v}_k = \mathbf{0}$ which can be explicitly written as

$$
\begin{aligned}
e_k^x v_k^y &= e_k^y v_k^x \\
e_k^y v_k^z &= e_k^z v_k^y \\
e_k^z v_k^x &= e_k^x v_k^z
\end{aligned}
\tag{4}
$$

Define $\overline{\mathbf{T}}$ as

$$
\overline{\mathbf{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
$$

Eq. (4) can be written as

$$
\operatorname{diag}(\mathbf{e}_k)\overline{\mathbf{T}}\mathbf{v}_k - \operatorname{diag}(\overline{\mathbf{T}}\mathbf{e}_k)\mathbf{v}_k = \mathbf{0}
\tag{5}
$$

where $\operatorname{diag}(\mathbf{x})$ is the diagonal matrix of which the ith diagonal component is equal to the ith component x_i of $\mathbf{x}$. By letting

$$
\mathbf{D}_k = \operatorname{diag}(\mathbf{e}_k)\overline{\mathbf{T}} - \operatorname{diag}(\overline{\mathbf{T}}\mathbf{e}_k)
$$

Eq. (5) can be rewritten as

$$
\mathbf{D}_k\mathbf{v}_k = \mathbf{0}
\tag{6}
$$

By assembling Eq. (6) through all members for which the directions are specified, the following linear relation is derived for $\mathbf{v}$:

$$
\mathbf{D}\mathbf{v} = \mathbf{0}
\tag{7}
$$

3.2. Directions of fixed members

In order to consider the fixed nodes (supports) in a similar manner as free nodes (internal nodes) in the self-equilibrium equation, we have introduced the concept of auxiliary fixed members, of which the directions are to be specified.

For a three-dimensional structure, let $\mathbf{X}_i = (x_i, y_i, z_i)^\top$ denote the coordinate vector of node i. The coordinate difference vector $\mathbf{d}_k = (d_k^x, d_k^y, d_k^z)^\top$ of member k that connects nodes i and j $(i < j)$ is defined as

$$
\mathbf{d}_k = \mathbf{X}_j - \mathbf{X}_i
\tag{8}
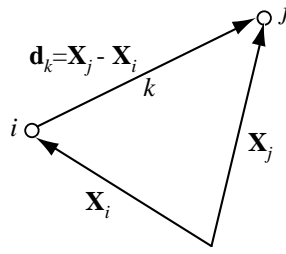$$

which is illustrated in Fig. 4.

Fig. 4. Coordinate difference vector $\mathbf{d}_k$ of member k ($i < j$).

Using the relation between the direction of a member and its force vector, $\mathbf{d}_k \times \mathbf{v}_k = \mathbf{0}$ should be satisfied; i.e.

$$
\begin{aligned}
d_k^x v_k^y &= d_k^y v_k^x \\
d_k^y v_k^z &= d_k^z v_k^y \\
d_k^z v_k^x &= d_k^x v_k^z
\end{aligned}
\tag{9}
$$

Since Eq. (9) has the same form as Eq. (4), the relation similar to Eq. (5) can be easily obtained as

$$
\operatorname{diag}(\mathbf{d}_k)\overline{\mathbf{T}}\mathbf{v}_k - \operatorname{diag}(\overline{\mathbf{T}}\mathbf{d}_k)\mathbf{v}_k = \mathbf{0}
\tag{10}
$$

$\mathbf{d}_k$ can be expressed as follows using Eq. (8):

$$
\mathbf{d}_k = -\mathbf{B}_k\mathbf{X}
\tag{11}
$$

where the rows of $\mathbf{B}_k \in \Re^{3 \times 3n}$ consist of the $(3k-2)$th, $(3k-1)$th, and $(3k)$th rows of $\mathbf{B}$ for a three-dimensional structure.

Let $\mathbf{d}^f \in \Re^{3n_f}$ denote the vector consisting of the coordinate difference vectors of the fixed members. The relation between $\mathbf{d}^f$ and $\mathbf{X}^f$ is written as

$$
\mathbf{d}^f = -\mathbf{B}^f\mathbf{X}^f
\tag{12}
$$

The vector consisting of force vectors of the fixed members is denoted by $\mathbf{v}^f \in \Re^{3m_f}$. Let $\mathbf{I}^f \in \Re^{m_f \times m_f}$ denote the identity matrix. By using $\mathbf{T}^f = \mathbf{I}^f \otimes \overline{\mathbf{T}}$, Eq. (10) for fixed members is assembled as

$$
\operatorname{diag}(\mathbf{d}^f)\mathbf{T}^f\mathbf{v}^f - \operatorname{diag}(\mathbf{T}^f\mathbf{d}^f)\mathbf{v}^f = \mathbf{0}
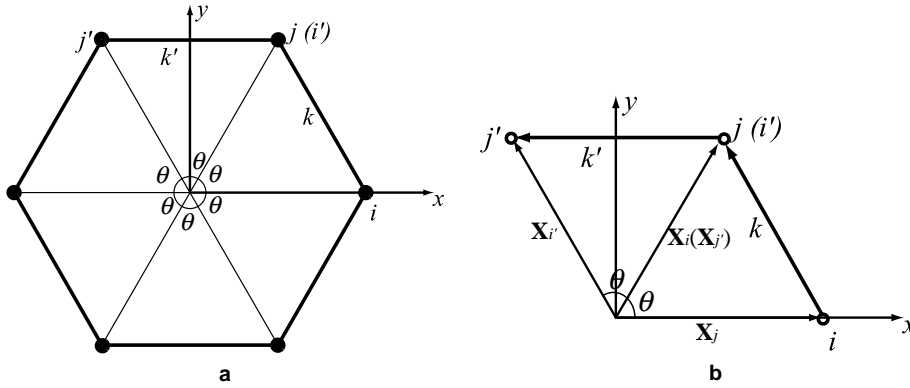\tag{13}
$$

In Eq. (13), $\mathbf{d}^f$ is known after the coordinates $\mathbf{X}^f$ of the fixed nodes are specified, and $\mathbf{T}^f$ is a constant matrix. Since $\mathbf{v}^f$ is the selected components of $\mathbf{v}$, it is easy to see that Eq. (13) can be rewritten by using a known matrix $\mathbf{C}$ as

$$
\mathbf{C}\mathbf{v} = \mathbf{0}
\tag{14}
$$

3.3. Symmetry properties

The configuration of a tensegrity system usually has symmetry properties; i.e. invariance conditions to reflection with respect to some planes and/or rotation around some axes. Therefore, the member direction vectors should be specified to satisfy such symmetry conditions. The same axial forces should be assigned to the symmetrically located members.

For example, consider a part of a two-dimensional structure shown in Fig. 5(a), whose members are rotationally arranged by $\theta \, (= \pi/3)$. Select two adjacent members k and k' as shown in Fig. 5(b) to illustrate

Fig. 5. Rotational symmetry property of a two-dimensional structure ($i < j$ and $i' < j'$): (a) configuration of a two-dimensional structure and (b) rotational symmetry of members k and k'.

the process of formulating their rotational symmetry properties. Members k and k' connect pairs of nodes (i,j) and (i',j'), respectively. The rotation matrix is defined as

$$\mathbf{M}_l = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then the relation between the coordinate difference vectors of members k and k' is written as

$$\mathbf{d}_{k'} = \mathbf{M}_l \mathbf{d}_k$$

From Figs. 1 and 4, we know that the direction of $\mathbf{d}_k$ is same as the positive direction of $\mathbf{v}_k$. So the symmetry property of axial forces of members k and k' can be written as

$$\mathbf{v}_{k'} = \mathbf{M}_l \mathbf{v}_k \tag{15}$$

By letting $\mathbf{S}_l = (\mathbf{0} \ \cdots \ \mathbf{M}_l \ \cdots \ -\mathbf{I} \ \cdots \ \mathbf{0})$, Eq. (15) can be rewritten as

$$\mathbf{S}_l \mathbf{v} = \mathbf{0} \tag{16}$$

The rotational symmetry of all other members of the structure can be formulated in a similar way. Reflectional symmetry can be also written in a similar form as Eq. (16). By combining Eq. (16) through all the symmetry conditions, the following linear equations are obtained:

$$\mathbf{S}\mathbf{v} = \mathbf{0} \tag{17}$$

4. Form-finding process

In this section, we introduce an algorithm for directly specifying the member axial forces and nodal coordinates consecutively. Linear equations are formulated based on the equilibrium equations and geometrical constraints in terms of the generalized force vector.

4.1. Axial forces

From the equilibrium conditions (3) and the geometrical constraints (7), (14) and (17), we obtain

$$\begin{pmatrix} \mathbf{B}^\top \\ \mathbf{D} \\ \mathbf{C} \\ \mathbf{S} \end{pmatrix} \mathbf{v} = \mathbf{0} \tag{18}$$

By letting $\mathbf{H}^\top = (\mathbf{B}, \mathbf{D}^\top, \mathbf{C}^\top, \mathbf{S}^\top)$, Eq. (18) can be rewritten as

$$\mathbf{Hv} = \mathbf{0} \tag{19}$$

At this stage, our task is to find a nontrivial solution $\mathbf{v} \neq \mathbf{0}$ to Eq. (19). Let $h = 3m - \text{rank}(\mathbf{H})$ for a three-dimensional structure. If $h = 0$, then there exists only trivial solution $\mathbf{v} = \mathbf{0}$. If $h > 0$, then the static relation (19) is underdetermined. Tensegrity systems will often fall into this category. So we will focus only on the underdetermined case in this paper.

The solution of Eq. (19) can be written by using a matrix $\mathbf{G} \in \mathfrak{R}^{3m \times h}$ as

$$\mathbf{v} = \mathbf{G}\boldsymbol{\alpha} \tag{20}$$

where the columns of $\mathbf{G}$ are self-equilibrium modes and the ith element α_i of $\boldsymbol{\alpha} \in \mathfrak{R}^h$ is the coefficient of the ith self-equilibrium mode. Since $\boldsymbol{\alpha}$ has no explicit mechanical meaning, we will obtain $\boldsymbol{\alpha}$ by specifying $\bar{\mathbf{v}}$ that consists of a set of independent components of generalized force vector.

Let $\mathscr{I} \subseteq \{1, \ldots, 3m\}$ denote the set of indices of components of $\mathbf{v}$ to be specified. $\bar{\mathbf{v}}$ is defined as the vector consisting of the component v_j ($j \in \mathscr{I}$) of $\mathbf{v}$. By assembling the corresponding rows of $\mathbf{G}$ to generate a sub-matrix $\bar{\mathbf{G}}$, the relation between $\bar{\mathbf{v}}$ and $\boldsymbol{\alpha}$ is written as

$$\bar{\mathbf{v}} = \bar{\mathbf{G}}\boldsymbol{\alpha} \tag{21}$$

If $\bar{\mathbf{G}} \in \mathfrak{R}^{h \times h}$ is full-rank, Eq. (21) can be solved as

$$\boldsymbol{\alpha} = \bar{\mathbf{G}}^{-1}\bar{\mathbf{v}} \tag{22}$$

By substituting $\boldsymbol{\alpha}$ back to Eq. (20), the force vector $\mathbf{v}$ of all members is obtained as

$$\mathbf{v} = \mathbf{G}\bar{\mathbf{G}}^{-1}\bar{\mathbf{v}} \tag{23}$$

Let $(\mathbf{G})_k$ denote the kth row vector of $\mathbf{G}$. $\Pi = \{\pi(l) | l = 1, 2, \ldots, 3m\}$ denotes a permutation of $3m$ indices $1, 2, \ldots, 3m$, where $\pi(l)$ stands for the location of index l in Π. The following algorithm generates $\mathscr{I}$ and $\bar{\mathbf{G}}$, where the reduced row-echelon form (RREF) (Borse, 1997) summarized in Appendix A is effectively used:

Algorithm 1

Step 0: Let $\mathscr{I} = \emptyset$, feasible set $\mathscr{A} = \{1, 2, \ldots, 3m\}$, $\Pi^0 = \{\pi^0(l) | l = 1, 2, \ldots, 3m\}$ and $\pi^0(l) = l$ ($l = 1, \ldots, 3m$). Set $i := 0$.
Step 1: If $i = h$, then $\bar{\mathbf{G}} := \hat{\mathbf{G}}$, and STOP. Otherwise, set $i \leftarrow i + 1$.
Step 2: Choose $j \in \mathscr{A}$. Update $\mathscr{I} := \mathscr{I} \cup j$. Define $\Pi^i = \{\pi^i(l) | l = 1, 2, \ldots, 3m\}$ by

$$\pi^i(l) = \begin{cases} \pi^{i-1}(l) & (l < j) \\ 3m & (l = j) \\ \pi^{i-1}(l) - 1 & (l > j) \end{cases}$$

Step 3: Generate $\mathbf{Q}$ by eliminating $(\mathbf{G})_k$ ($\forall k \in \mathscr{I}$) from $\mathbf{G}$. Let $\hat{\mathbf{G}}$ be the matrix consisting of $(\mathbf{G})_k$ ($\forall k \in \mathscr{I}$).

Step 4: Compute the RREF of the matrix $(\hat{\mathbf{G}}^\top, \mathbf{Q}^\top)$ in a form of

$$\mathbf{W} = \begin{pmatrix} \mathbf{I} & \mathbf{W}^U \\ \mathbf{O} & \mathbf{W}^L \end{pmatrix}$$

where $\mathbf{W}^U \in \Re^{i \times (3m-i)}$ and $\mathbf{W}^L \in \Re^{(h-i) \times (3m-i)}$.

Step 5: Update $\mathscr{A}$ as

$$\mathscr{A} = \{l \mid (\mathbf{W}^L)_{\pi^i(l)} \neq \mathbf{0} \ (\pi^i(l) = 1, \ldots, 3m - i)\}$$

and go to Step 1.

4.2. Nodal coordinates

Knowing only the generalized force vector $\mathbf{v}$, we are still unable to determine configuration uniquely. Since $\mathbf{v}_k$ for all members are known by using the procedure presented in Section 4.1, Eq. (10) is rewritten as

$$\text{diag}(\overline{\mathbf{T}}\mathbf{v}_k)\mathbf{d}_k - \text{diag}(\mathbf{v}_k)\overline{\mathbf{T}}\mathbf{d}_k = \mathbf{0} \tag{24}$$

Let $\mathbf{I} \in \Re^{m \times m}$ denote the identity matrix. By using $\mathbf{T} = \mathbf{I} \otimes \overline{\mathbf{T}}$, Eq. (24) is assembled through all members as

$$\text{diag}(\mathbf{T}\mathbf{v})\mathbf{d} - \text{diag}(\mathbf{v})\mathbf{T}\mathbf{d} = \mathbf{0} \tag{25}$$

where $\mathbf{d} = (\mathbf{d}_1^\top, \ldots, \mathbf{d}_m^\top)^\top$.

Incorporating Eq. (11) into Eq. (25), the constraints on $\mathbf{X}$ can be written in the following form:

$$\mathbf{F}\mathbf{X} = \mathbf{0} \tag{26}$$

where

$$\mathbf{F} = \text{diag}(\mathbf{T}\mathbf{v})\mathbf{B} - \text{diag}(\mathbf{v})\mathbf{T}\mathbf{B}$$

is a known matrix. Since the same axial forces have been assigned to the symmetrically located members, the symmetry conditions (17) are included in Eq. (26).

Let $\gamma = 3n - \text{rank}(\mathbf{F})$ and suppose an underdetermined case $\gamma > 0$. The solution of Eq. (26) can be written as

$$\mathbf{X} = \mathbf{P}\boldsymbol{\beta} \tag{27}$$

where $\boldsymbol{\beta} \in \Re^\gamma$ is the coefficient vector and $\mathbf{P} \in \Re^{3n \times \gamma}$. The nodal coordinates are divided into unknown components $\mathbf{X}^c \in \Re^{3n_c}$ of the free nodes and specified components $\mathbf{X}^f \in \Re^{3n_f}$ of the fixed nodes (supports). The matrix $\mathbf{P}$ is divided into $\mathbf{P}^c$ and $\mathbf{P}^f$, accordingly. Hence, Eq. (27) is rewritten as

$$\begin{pmatrix} \mathbf{X}^c \\ \mathbf{X}^f \end{pmatrix} = \begin{pmatrix} \mathbf{P}^c \\ \mathbf{P}^f \end{pmatrix} \boldsymbol{\beta} \tag{28}$$

Let $r_f = \text{rank}(\mathbf{P}^f)$. Select r_f independent rows from $\mathbf{P}^f$ to obtain matrix $\overline{\mathbf{P}}$ utilizing RREF forms. The vector $\overline{\mathbf{X}}$ of independent nodal coordinates are selected from $\mathbf{X}^f$ accordingly. If $r_f = \gamma$, we can calculate the coordinate vector $\mathbf{X}$ directly by

$$\mathbf{X} = \mathbf{P}\overline{\mathbf{P}}^{-1}\overline{\mathbf{X}} \tag{29}$$

Otherwise, we are able to specify $(\gamma - r_f)$ nodal coordinates to obtain $\overline{\mathbf{X}}$ by using the same procedure described in Section 4.1.

4.3. Stress states of members

For a tensegrity system, it is important to know whether each member is in tension or in compression. From Figs. 1 and 4, we can see that the direction of $\mathbf{d}_k$ is same as that of the member force vector in tension. So the inner product g_k of $\mathbf{d}_k$ and $\mathbf{v}_k$ have the following properties:

$$g_k = \mathbf{v}_k^\top \mathbf{d}_k \quad \begin{cases} > 0 & \text{tension} \\ < 0 & \text{compression} \\ = 0 & \text{member } k \text{ is removable} \end{cases} \tag{30}$$

For the case of $g_k = 0$, member k can be removed because

(a) If $\mathbf{v}_k = \mathbf{0}$, then there exists no force in member k, and its existence is unnecessary.
(b) If $\mathbf{d}_k = \mathbf{0}$, then nodes i and j coincide. So member k and node i or j can be removed.

4.4. Design procedure of tensegrity systems

In Sections 4.1 and 4.2, we have presented the procedure and algorithm for designing a tensegrity system. In order to show how to design a tensegrity system systematically, we summarize the design procedure as follows:

Design procedure:

Step 1: Generate the self-equilibrium system by replacing the supports with the auxiliary fixed members.
Step 2: Give the topology, represented by the incidence matrix $\mathbf{B}$.
Step 3: Assign the directions of some members, and coordinates of supports so as to define $\mathbf{D}$ and $\mathbf{C}$, respectively. Give the symmetry properties by the matrix $\mathbf{S}$.
Step 4: Specify $\bar{\mathbf{v}}$ and obtain $\overline{\mathbf{G}}$ by using Algorithm 1. Compute $\mathbf{v}$ from Eq. (23).
Step 5: Specify $\overline{\mathbf{X}}$ and obtain $\overline{\mathbf{P}}$ by using the algorithm described in Section 4.2. Compute $\mathbf{X}$ from Eq. (29).
Step 6: Remove member k satisfing $\mathbf{v}_k^\top \mathbf{d}_k = 0$. Convert the auxiliary fixed members back to supports.

5. Numerical examples

In this section, a simple two-dimensional and a rotationally symmetric three-dimensional prestressed tensegrity systems are presented firstly to show how the auxiliary fixed members can be used effectively. Diamond-shaped and prism-shaped tensegrity systems are then investigated to demonstrate the capability of the proposed method for generating various shapes.

5.1. A simple two-dimensional prestressed tensegrity system

The initial topology of a two-dimensional structure in this example is shown in Fig. 2. It may be reduced from a three-dimensional structure by using some of its geometrical characteristics; e.g. symmetry properties (Ohsaki and Kanno, 2003).

We have discussed in Section 2 that the supports can be converted into free nodes by using the concept of auxiliary fixed member in order to obtain the desirable configuration. After implementation of form-finding

using the method proposed in Section 4, the auxiliary fixed member may be removed to convert the two nodes connected by it back to the supports.

To illustrate the procedure of specifying the member directions, member 5 is assigned to be vertical. The nodal coordinates of the two supports are specified as $(0,0)$ and $(7,0)$. By connecting the two supports with auxiliary fixed member 9, the new topology of the structure is shown in Fig. 3. After defining the directions of members 5 and 9, the rank of matrix $\mathbf{H}$ has been computed to obtain $h = 6$; i.e. we are able to specify 6 axial force components.

As Example 1, if we specify $\bar{\mathbf{v}} = (\mathbf{v}_2, \mathbf{v}_4, v_5^y, v_6^x)^\top = ((1,0),(0,-1),1,-1)^\top$ consecutively based on Algorithm 1, the axial force components of all the members are obtained as $\mathbf{v} = ((-1,-1),(1,0),$ $(1,-1),(0,-1),(0,1),(-1,1),(1,0),(1,1),(-2,0))^\top$.

The rank of matrix $\mathbf{F}$ is computed to obtain $\gamma - r_f = 4 - 3 = 1$. So we can specify one nodal coordinate except those of the defined fixed nodes 5 and 6. If $X_1^x = 2$, the structure as shown in Fig. 6(a) is obtained, where the thick lines represent members in compression, the thin lines are in tension, and the auxiliary fixed member 9 has been removed. If $X_2^y = 1.5$ as Example 2, we will achieve the configuration in Fig. 6(b).

The direction of member 4 is not necessarily vertical. If we specify $\mathbf{v}_4 = (0.5, 1)^\top$ as Example 3, and specify other variables the same as in Example 2, the configuration as shown in Fig. 6(c) is obtained. This way, the directions of the struts can be easily controlled based on the designer's preference.

5.2. A three-dimensional prestressed tensegrity system

The three-dimensional prestressed tensegrity system shown in Fig. 7(a) consists of 24 free nodes, 8 fixed nodes and 60 members. Its fixed nodes are located on a circle, the radius of which is 15 m. Auxiliary fixed members, which are shown in dashed lines in Fig. 7(b), are utilized to substitute the fixed nodes to free nodes, and to transform the prestressed structure into a self-stressing structure. Therefore, there are 66 members but no fixed node in the substituted model.
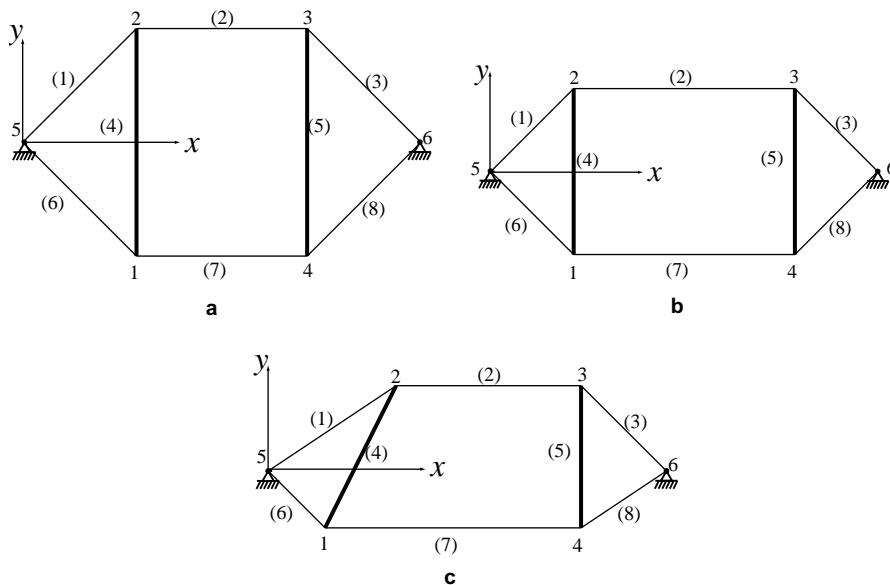

Fig. 6. Two-dimensional tensegrity system: (a) Example 1, (b) Example 2 and (c) Example 3.
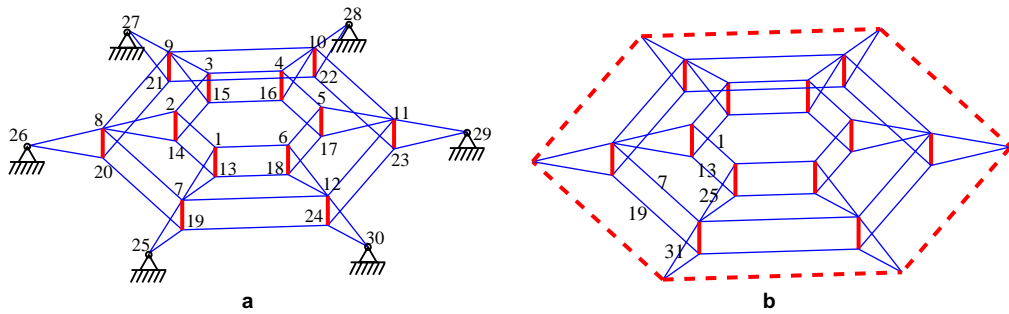
Fig. 7. Perspective view of a three-dimensional prestressed tensegrity system: (a) initial model and (b) substituted model.

Without introducing any geometrical or mechanical constraint, there are totally $(h =)99$ independent components of axial forces that can be specified arbitrarily. This may be thought as a burden rather than benefit since a large number of axial forces have to be specified by designers. Since the structure used as an architecture usually has symmetric properties, we classify its cables into eight groups; six cables in each group. The cables in each group are rotationally symmetric around z-axis by $\pi/3$. This way, there are only $(h =)15$ independent components of axial forces needed to be specified.

As Example 1, we specify 15 independent axial forces as $\bar{\mathbf{v}}^\top = (\mathbf{v}_1, \mathbf{v}_7, v_{13}^x, v_{13}^y, v_{19}^x, \mathbf{v}_{25}, \mathbf{v}_{31}) = ((-1, 1.7321, 0), (-1, 1.7321, 0), -1, 1.7321, -3, (-1, -1.7321, -0.5), (-3, -5.1963, -1.5))$ as indicated in Fig. 7(b). The number of independent nodal coordinates that can be specified is calculated to be $\gamma - r_f = 2$ in this case. If we specify the x-coordinates of nodes 2 and 8 as -5 and -10, respectively, we can obtain configuration of the structure as shown in Fig. 7(a) and Fig. 8, where the auxiliary fixed members have been removed.

Consider Example 2, where we change the values of v_{31}^x and v_{31}^y in Example 1 to -2 and 3.4642, respectively, without changing the values of other parameters, we can achieve a configuration as shown in Fig. 9.

5.3. Diamond-shaped tensegrity

Consider a tensegrity system that consists of 6 nodes and 13 members as shown in Fig. 10. The configurations are found for the following cases:
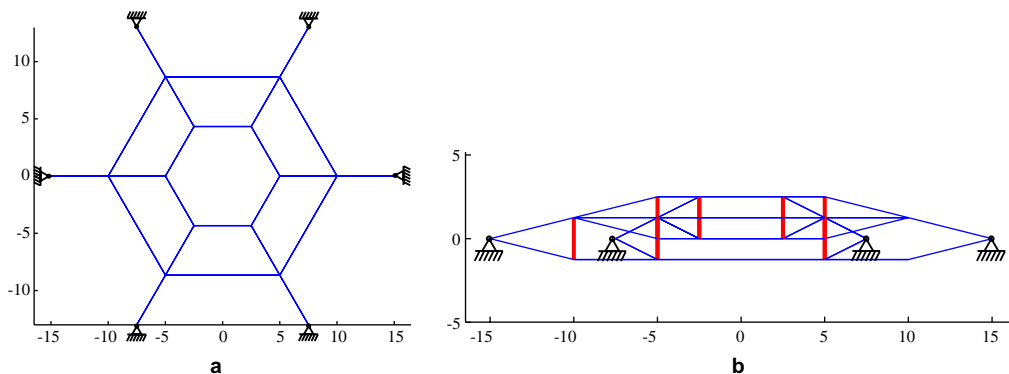

Fig. 8. Example 1 of the three-dimensional prestressed tensegrity system: (a) top view and (b) side view.
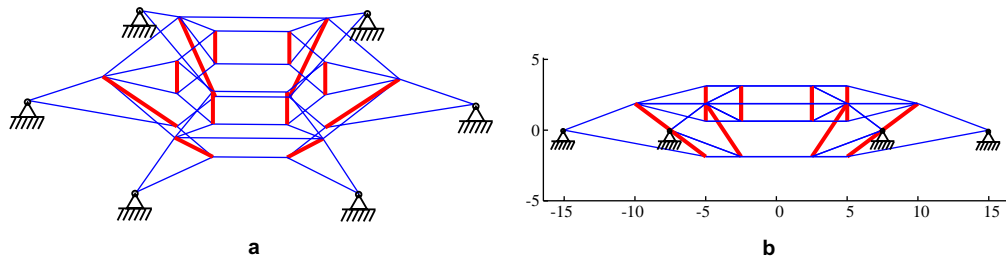
Fig. 9. Example 2 of the three-dimensional prestressed tensegrity system: (a) perspective view and (b) side view.
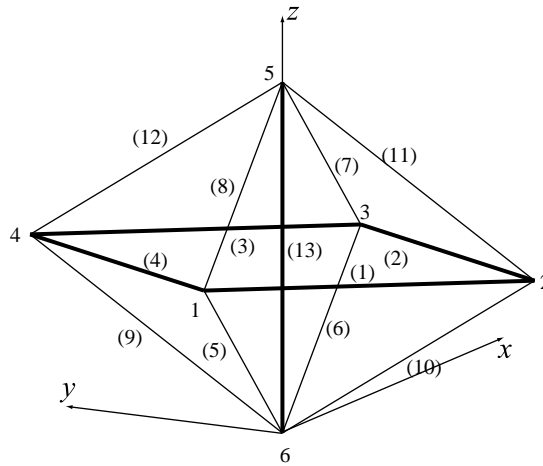

Fig. 10. Example 1 of diamond-shaped tensegrity.

Consider firstly Example 1 without any explicit geometrical constraint. The rank of matrix $\mathbf{H}$ has been computed to find $h = 22$; i.e. there are 22 axial force components needed to be specified. By using the algorithm presented in Section 4.1, we specify the forces in members 5, 6, 7 and 8 as shown in *Variables* in Table 1. The feasible set obtained by using the RREF form is $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}, \mathbf{v}_{12}, v^z_{13})$. The axial forces $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_9, v^z_{10})$ have been selected from the feasible set consecutively by using Algorithm 1 as shown in *Variables* in Table 1. Then the results computed by Eq. (23) are shown in *Results*.

The rank of matrix $\mathbf{F}$ has been computed to find $\gamma = 4$; i.e. there exist four nodal coordinates needed to be specified. The specified four nodal coordinates and the results using a method similar to Algorithm 1 for specifying force components are shown in Table 2. The obtained configuration is as shown in Fig. 10. Note

Table 1
Axial forces of Example 1 of diamond-shaped tensegrity

	Variables $\mathbf{v}_k$								Results $\mathbf{v}_k$					
k	1	2	5	6	7	8	9	10	3	4	10	11	12	13
x	1	1	−1	1	1	−1	0		−1	1	0	0	0	0
y	−1	1	0	0	0	0	1		1	1	−1	−1	1	0
z	0	0	1	1	−1	−1	1	1	0	0		−1	−1	−4

Table 2
Nodal coordinate of Example 1 of diamond-shaped tensegrity

	Variables $\mathbf{X}_i$		Results $\mathbf{X}_i$				
i	5	6	1	2	3	4	5
x		0	−2	0	2	0	0
y		0	0	−2	0	2	0
z	4	0	2	2	2	2	

that the locations and force vectors of members 1–4 have been obtained to be rotationally symmetric around z-axis by $\pi/2$, although only the force vectors of members 1 and 2 have been specified.

If we specify $v_{10}^z = 3$, which is different from $v_{10}^z = 1$ in the previous example, then all nodes will be degenerated into one node with only three nodal coordinates that can be specified; i.e. $\gamma = 3$. It means that we cannot obtain the desirable configuration although the generalized force vector $\mathbf{v}$ satisfies the equilibrium conditions and all geometrical constraints.

In the following examples of the diamond-shaped tensegrity system, we will show how to search new configuration practically by changing the values of some variables to be specified. To reduce the number of independent variables or to assign geometrical characteristics, we introduce some explicit geometrical constraints such that members 1–4 are symmetrically located around z-axis by $\pi/2$, and member 13 is chosen as a fixed member; i.e. the nodal coordinates of nodes 5 and 6 are given as $(0,0,4)$ and $(0,0,0)$, respectively. In this case, there are only 13 components of axial forces needed to be specified; i.e. $h = 13$, and no nodal coordinate can be given because $r_f = \gamma = 4$.

Consider Example 2 with the symmetric geometrical constraints as described above. In this example, we specify $(\mathbf{v}_1)_{e2} = 2(\mathbf{v}_1)_{e1}$, where $(\mathbf{v}_1)_{ei}$ denote the axial force vector of member 1 in Example i. The axial forces of members 5, 6, 9 and 10 are same as those in Example 1. The specified variables and computed results of axial forces are shown in Table 3. The results of nodal coordinates are listed in Table 4.

We can see from Fig. 11 that the compressive element consisting of symmetrically arranged members 1–4 is located at a higher place than in Example 1, because larger values have been given for the force components of member 1.

If we let $(\mathbf{v}_1)_{e3} = 0.2(\mathbf{v}_1)_{e2}$ and the other variables of axial forces remain the same as those in Example 2, then we obtain a new configuration as shown in Fig. 12 as Example 3.

Table 3
Axial forces of Example 2 of diamond-shaped tensegrity

	Variables $\mathbf{v}_k$					Results $\mathbf{v}_k$								
k	1	5	6	9	10	2	3	4	7	8	10	11	12	13
x	2	−1	1	0		2	−2	2	3	−3	0	0	0	0
y	−2	0	0	1		2	2	2	0	0	−1	−3	3	0
z	0	1	1	1	1	0	0	0	−1	−1		−1	−1	−4

Table 4
Nodal coordinates of Example 2 of diamond-shaped tensegrity

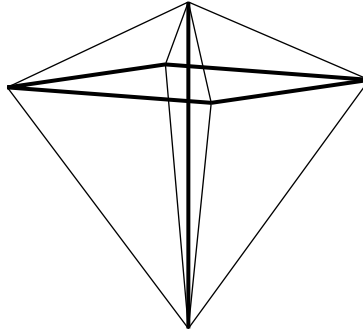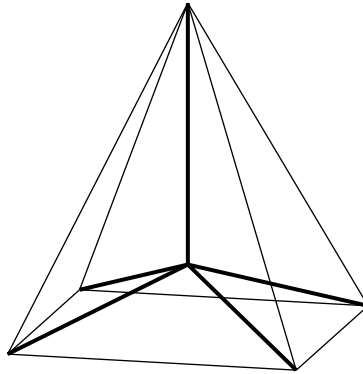|	Variables $\mathbf{X}_i$		Results $\mathbf{X}_i$			
i	5	6	1	2	3	4
x	0	0	−3	0	3	0
y	0	0	0	−3	0	3
z	4	0	3	3	3	3

Fig. 11. Example 2 of diamond-shaped tensegrity.

Fig. 12. Example 3 of diamond-shaped tensegrity.

If we define the symmetry property such that the members 7, 8, 11 and 12 are symmetrically located with members 6, 5, 10 and 9, respectively, as shown in Fig. 10, instead of rotational symmetry, then there exist only seven components of the generalized force vector to be specified.

5.4. Prism-shaped tensegrity

Consider a prism-shaped structure consisting of 6 nodes and 12 members as shown in Fig. 13. Member sets {1,2,3}, {7,8,9} and {10,11,12} are arranged, respectively, in rotationally symmetric locations. The rank of **H** has been computed to find that there are seven axial force components to be specified.

If we specify $(v_1^x, v_1^y, v_1^z, v_4^z, v_9^x, v_9^y, v_9^z)$ consecutively to be $(0.866, -0.5, 0, 1, 1, 0, 0)$ for $\bar{\mathbf{v}}^\top$ by using Algorithm 1 as Example 1, then we can specify four nodal coordinates by computing the rank of **F**. By specifying $(x_1, y_1, z_1, x_2) = (0, 0, 0, 4)$ for $\overline{\mathbf{X}}^\top$, we obtain a configuration as shown in Fig. 13.

Let $v_9^y = 2$ and $v_9^y = 0.5$ for Examples 2 and 3, respectively, which are equal to twice and half of the value in Example 1. The configurations for Examples 2 and 3 are obtained as shown in Figs. 14 and 15, respectively, where the sizes of the top and bottom triangles are different.

5.5. Evaluation of design

If the tangent stiffness matrix of a tensegrity structure is positive-definite, when its rigid-body motions are constrained, then the structure is stable. Using this criterion, stabilities of the examples shown above have been confirmed by checking eigenvalues of tangent stiffness matrix of the structure (Murakami, 2001).
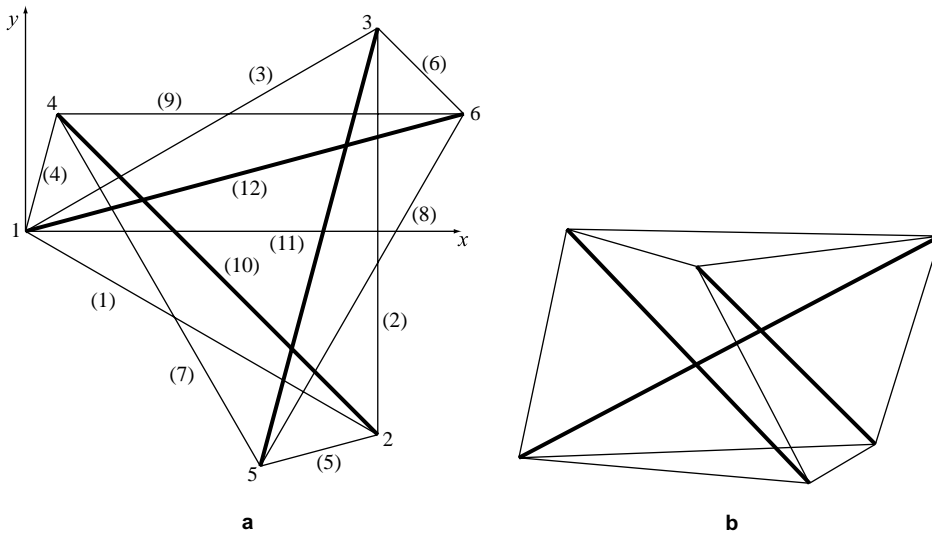
Fig. 13. Example 1 of rotationally symmetric prism-shaped tensegrity: (a) top view and (b) side view.
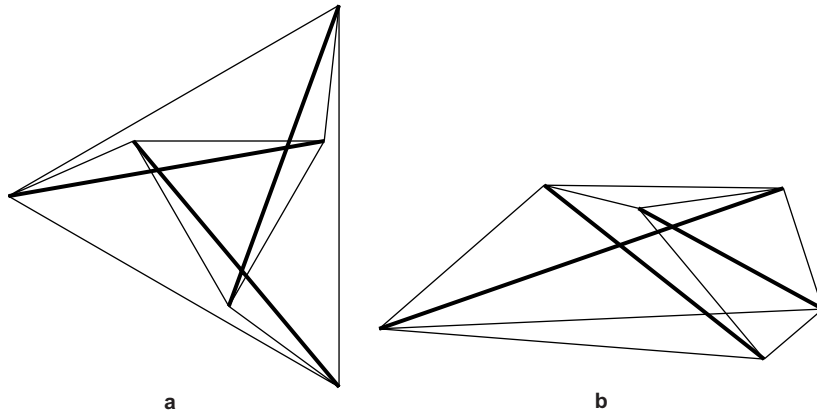

Fig. 14. Example 2 of rotationally symmetric prism-shaped tensegrity with $v_9^y = 2$: (a) top view and (b) side view.

By using Eqs. (19) and (26), the errors e_f and e_c of force vectors and nodal coordinates, respectively, for a three-dimensional structure are defined as

$$e_f = \sqrt{(\mathbf{H}\mathbf{v})^\top \mathbf{H}\mathbf{v}/(3m)} \tag{31}$$

$$e_c = \sqrt{(\mathbf{F}\mathbf{X})^\top \mathbf{F}\mathbf{X}/(3n)} \tag{32}$$

For each example given in Section 5, the calculation errors of forces and nodal coordinates are within 10^{-15} and 10^{-14}, respectively, by using Eqs. (31) and (32), which confirms the accuracy of the proposed method.
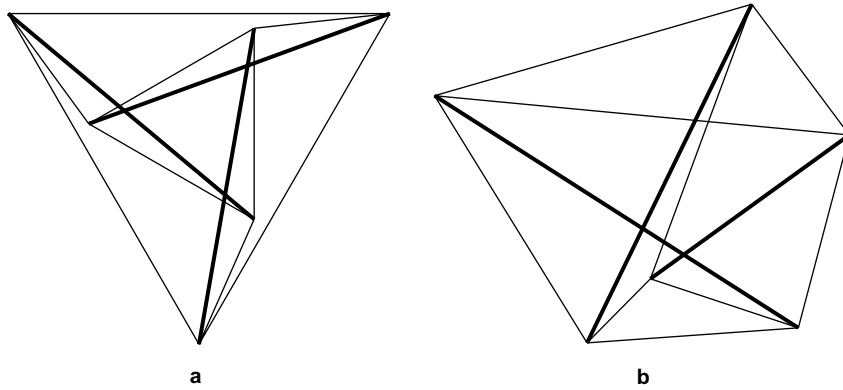
Fig. 15. Example 3 of rotationally symmetric prism-shaped tensegrity with $v_9^y = 0.5$: (a) top view and (b) side view.

6. Discussions and conclusions

A general method has been presented for direct design of member directions, internal forces and nodal locations of tensegrity systems with given topology, which is defined as a directed graph. The system of equilibrium equations is written in terms of the components of the member force vectors by using the incidence matrix.

The directions of members and symmetry properties are first assigned as geometrical constraints, and the member force vectors are computed from the constrained equilibrium equations. The locations of some nodes including the supports are then assigned to obtain the locations of all nodes. A concept of auxiliary (fixed) members is introduced to present a unified approach for tensegrity systems with or without supports.

The solution obtained by this method satisfies the equilibrium conditions and the geometrical constraints exactly. By using the proposed method, designers can directly control the axial forces and the configuration simultaneously, which is considered to be a major advantage of the method. New configurations can also be obtained by changing the forces and geometrical constraints.

The proposed method is very efficient because only linear equations needed to be solved. A general algorithm has been presented to find the independent variables consecutively. Unfortunately, it is not an easy task to specify the members to be in tension or compression because the equilibrium shape is not known a priori.

In the case where the number of independent variables that should be specified is large, the geometrical constraints including the symmetry properties should be extensively used. In some cases, the obtained configuration may not be feasible, which means that some nodes may contact and/or some members may intersect with each other. For these cases, the variables to be specified should be gradually varied from a feasible solution. Introducing more geometrical constraints may be another way to prevent infeasible solutions.

Appendix A. Reduced row-echelon form (RREF)

Any (possibly not square) finite matrix $\mathbf{A}$ can be reduced by a finite sequence of linear elementary row operation $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_l$, each of which invertible, to RREF $\mathbf{U} := \mathbf{E}_m \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A}$ characterized by the following three properties:

(a) The first nonzero element in any nonzero row is 1.
(b) The leading 1 of each nonzero row 1 appears in a column of which all the other elements are 0.

(c) Each such leading 1 comes in a column after every preceding row's leading zeros.

For example, the RREF of matrix

$$\mathbf{A} = \begin{pmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{pmatrix}$$

is

$$\mathbf{U} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ \hline 0 & 0 & 0 & 0 \end{array} \right)$$

It can be proved that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{U})$. We can easily see that the rank of $\mathbf{U}$ is 3. From the fact that we have applied only row operations on $\mathbf{A}$ to get $\mathbf{U}$, we can know from $\mathbf{U}$ that the first three columns of $\mathbf{A}$ are independent.

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